Airport Walkway Problem

Solutions by Olen Rambow, December 16, 2008

Statement of the Problem:

Suppose you are trying to get from one end A of an airport terminal to the other end B. (For simplicity, assume the terminal is a one-dimensional line segment.) Some portions of the terminal have moving walkways; other portions do not. Your walking speed is a constant v, but while on a walkway, it is boosted by the speed u of the walkway for a net speed of v+u. Your objective is to get from A to B in the shortest time possible.

1. Suppose you need to pause for some period of time Δt, say to tie your shoe. Is it more efficient to do so while on a walkway, or off the walkway? Assume the period of time required is the same in both cases.

2. Suppose you have a limited amount of energy available to run at an increased speed of v' (or v'+u, if you are on a walkway). That is, you are able to run at speed v′ for time Δt, and the rest of the time you walk at speed v. Is it more efficient to run while on a walkway, or off the walkway? (That is, which way will shorten your total travel time more?)

3. Do the answers to the above questions change if one takes into account the various effects of special relativity? (This is of course an academic question rather than a practical one. But presumably it should be the time in the airport frame that one wants to minimize, not time in one’s personal frame.)

Solutions:

Let x be the total distance that must be walked on the floor, and let y be the total distance over which the walkways carry you; v is your walking speed, and u is the speed of the walkway with respect to the floor.

To get a feel for the problem, let's first calculate total travel time if you don't tie your shoe: You travel a distance x at speed v, which takes you a time x/v; and you travel a distance y at speed u+v, which takes you a time y/(u+v). Therefore the total time it takes you to travel the whole distance x+y is:

$$T = \frac{x}{v} + \frac{y}{u+v}$$

Solution to Part 1:

For this problem, let Δt be the time that it takes you to tie your shoe.
If you tie your shoe on the floor, the total travel time is:

$$* \quad T_{Floor} = \frac{x}{v} + \frac{y}{u+v} + \Delta t$$

(This is because you travel a distance x at speed v, and a distance y at speed u+v, and for time Δt you're sitting still.)

If you tie your shoe on the walkway, then for a time Δt you are moving at speed u (because you're just moving along with the walkway), so you travel a distance of uΔt while tying your shoe. This accounts for part of the length of the walkway (y). The remaining length of the walkway is then y – uΔt, and you travel that distance at a speed of u+v, which takes time (y – uΔt)/(u+v). Adding up the time spent on the floor, the time spent walking on the walkway, and the time spent tying your shoe on the walkway, you get:

$$T_{Walkway} = \frac{x}{v} + \frac{y - u\Delta t}{u+v} + \Delta t$$

This can be written differently in order to compare it to the total travel time for when you tie your shoe on the floor:

$$* \quad T_{Walkway} = \frac{x}{v} + \frac{y}{u+v} + \Delta t - \frac{u\Delta t}{u+v}$$

The last term is the only difference. Since it is negative, it is clear that if you tie your shoe on the walkway, the total travel time is less by an amount of uΔt/(u+v).

Solution for Problem 1:

Taking the difference between T_{Floor} and $T_{Walkway}$, we get:

$$T_{Floor} - T_{Walkway} = \frac{u\Delta t}{u+v}$$

You will save this much time by tying your shoe on the walkway instead of on the floor.

Solution to Part 2:

Here let Δt be the time that you spend running at speed v'.

If you choose to run while you're on the floor, then you travel a distance $v'\Delta t$ at speed v'. This takes up part of the distance that you must go on the floor (x). The remaining distance on the floor is then $x - v'\Delta t$, which you travel at a speed of v. While on the walkway, you travel a distance of y at a speed of $u+v$. Adding up the times for each of these parts, we get:

$$* \quad T_{Floor} = \Delta t + \frac{x - v'\Delta t}{v} + \frac{y}{u+v}$$

If you choose to run on the walkway, then you travel the whole floor distance x at a speed of v, which will take time x/v. While running, you travel part of the walkway distance at a speed of $u+v'$, and since you do so over a period of Δt, the distance covered while running is $(u+v')\Delta t$. The remaining distance on the walkway is then $y - (u+v')\Delta t$, which you travel at speed $u+v$. Adding these times up, we get:

$$* \quad T_{Walkway} = \Delta t + \frac{x}{v} + \frac{y-(u+v')\Delta t}{u+v}$$

These can both be rewritten as:

$$T_{Floor} = \Delta t + \frac{x}{v} + \frac{y}{u+v} - \frac{v'}{v}\Delta t \quad \text{and} \quad T_{Walkway} = \Delta t + \frac{x}{v} + \frac{y}{u+v} - \frac{u+v'}{u+v}\Delta t$$

Since all variables are positive and $v'>v$, we have

$$\frac{v'}{v} > \frac{u+v'}{u+v}$$

and therefore, more time is saved by running on the floor than by running on the walkway.

Solution for Problem 2:

Taking the difference between $T_{Walkway}$ and T_{Floor}, we get:

$$T_{Walkway} - T_{Floor} = \left[\frac{v'}{v} - \frac{u+v'}{u+v}\right]\Delta t$$

This is the amount of time that you save by running on the floor instead of running on the walkway. Interestingly, it can be rewritten as:

$$T_{Walkway} - T_{Floor} = \left(\frac{v'}{v} - 1\right)\frac{u\Delta t}{u+v}$$

In this second form, note that the factor on the right is the amount of time saved in problem 1 by tying your shoe on the walkway instead of on the floor. By examining the factor on the left, we see that if $v' = v$, the whole expression becomes zero, which is what we expect—if your running speed is the same as your walking speed, you don't save any time at all because the two situations become the same.

The conclusion is that it's better to run on the floor than on the walkway.

Solutions to Part 3:

Assumptions: When we say that it takes time Δt for you to tie your shoe, we mean that this is how much time it takes **in your reference frame**. If you are moving with respect to the airport's reference frame, the amount of time it takes you to tie your shoe from the airport's point of view will be greater. (Since you're moving, time slows down for you, so less time passes for you than for people sitting still in the airport watching you.) We also assume that you want to minimize the travel time from the airport's point of view (since it's their clock that determines when planes take off).

For problem 1:

If you tie your shoe on the floor, then when you tie your shoe, you are sitting still, so the time Δt that passes is the same for you and for the airport. When you are traveling on the floor at your walking speed v with respect to the airport, people in the airport see you moving at speed v across a distance x, so from their point of view it takes you a time x/v to travel along the floor. The only difference here is when you're walking on the walkway because velocities add differently in relativity. Instead of u+v, people in the airport will see you moving at a relativistic speed of:

$$v_{rel} = \frac{u+v}{1+\frac{uv}{c^2}}$$

Thus, to them it will appear that you travel a distance y at this speed.
Adding up all these times, we get:

$$T_{Floor} = \frac{x}{v} + \frac{y}{v_{rel}} + \Delta t$$

which, when expanded, becomes:

$$* \quad T_{Floor} = \frac{x}{v} + y\frac{1+\frac{uv}{c^2}}{u+v} + \Delta t$$

Now for the case in which you tie your shoe on the walkway. In the airport reference frame, you travel the whole floor distance x at speed v, which takes time x/v. You pause on the walkway for a time Δt in your reference frame, but you're traveling at speed u with respect to the airport, so to people in the airport you appear to be tying your shoe on the walkway for a time of γΔt, where γ is the Lorentz factor. Thus, to people in the airport, you spend this much time tying your shoe on the walkway:

$$\Delta t_{rel} = \frac{\Delta t}{\sqrt{1-\frac{u^2}{c^2}}}$$

Since you're traveling at speed u while tying your shoe, you will travel a distance $u\Delta t_{rel}$ during that time (according to the people in the airport). Then you travel the rest of the walkway distance of $y - u\Delta t_{rel}$ at your relativistic speed. Adding up these times, we get:

$$T_{Walkway} = \frac{x}{v} + \frac{y - u\Delta t_{rel}}{v_{rel}} + \Delta t_{rel}$$

Fully expanding these relativistic times and velocities gives us:

$$* \quad T_{Walkway} = \frac{x}{v} + y\frac{1+\frac{uv}{c^2}}{u+v} - u\Delta t\frac{(1+\frac{uv}{c^2})}{(u+v)\sqrt{1-\frac{u^2}{c^2}}} + \frac{\Delta t}{\sqrt{1-\frac{u^2}{c^2}}}$$

Final (Relativistic) Solution for Problem 1:

Taking the difference between T_{Floor} and $T_{Walkway}$, we get:

$$T_{Floor} - T_{Walkway} = \frac{u\Delta t}{u+v} + \left[1-\sqrt{1-\frac{u^2}{c^2}}\right]\frac{v\Delta t}{u+v}$$

Note that the first term is the same as the solution for the non-relativistic case.

Now examine the second term. Notice that when u is small, the second term vanishes, in which case the solution is the same as in the non-relativistic case. At the other extreme, as u approaches c, the second term approaches [vΔt/(u+v)], in which case the total difference becomes Δt. Thus, if the walkway is traveling at the speed of light, you don't lose any time at all by stopping to tie your shoe on it.

Our conclusion: Relativity actually **enhances** the time savings that you were already achieving by tying your shoe on the walkway instead of on the floor.

For Problem 2:

If you run while on the floor, people in the airport will see you moving at a speed of v′. To you, a time Δt passes; to them, it will be a longer time:

$$\Delta t_{rel} = \frac{\Delta t}{\sqrt{1-\frac{(v')^2}{c^2}}}$$

You cover a distance on the floor of $v'\Delta t_{rel}$ at speed v′ (according to the people in the airport). This leaves a floor distance of $x - v'\Delta t_{rel}$ to cover at your normal speed of v, which will take a time $(x - v'\Delta t_{rel})/v$ according to the people in the airport.

On the walkway, you travel a distance y at the speed obtained by adding u and v relativistically as before.

Adding all of these times up gives:

$$T_{Floor} = \Delta t_{rel} + \frac{x - v'\Delta t_{rel}}{v} + \frac{y}{v_{rel}}$$

Expanding this will give:

$$* \quad T_{Floor} = \frac{\Delta t}{\sqrt{1-\frac{(v')^2}{c^2}}} + \frac{x}{v} - \frac{v'}{v}\frac{\Delta t}{\sqrt{1-\frac{(v')^2}{c^2}}} + y\frac{1+\frac{uv}{c^2}}{u+v}$$

Now for the case in which you run on the walkway. The people in the airport see you moving at speed v while you walk on the ground a distance x, which takes time x/v.

When you're running on the walkway, your speed will be the relativistic sum of u and v′, which is:

$$v'_{rel} = \frac{u+v'}{1+\frac{uv'}{c^2}}$$

Time Δt passes for you, so a different time will pass for the people watching you:

$$\Delta t'_{rel} = \frac{\Delta t}{\sqrt{1-\frac{(v'_{rel})^2}{c^2}}}$$

While you are running on the walkway, the people will see you travel a distance of $v'_{rel}\Delta t'_{rel}$. So the remaining distance that you walk on the walkway will be $y - v'_{rel}\Delta t'_{rel}$. People see you traveling this distance at a speed of

$$v_{rel} = \frac{u+v}{1+\frac{uv}{c^2}}$$

Adding all these times up, we get:

$$T_{Walkway} = \Delta t'_{rel} + \frac{x}{v} + \frac{y - v'_{rel}\,\Delta t'_{rel}}{v_{rel}}$$

Expanding this yields:

$$* \quad T_{Walkway} = \frac{\Delta t}{\sqrt{1-\frac{(v'_{rel})^2}{c^2}}} + \frac{x}{v} + y\frac{1+\frac{uv}{c^2}}{u+v} - \frac{1+\frac{uv}{c^2}}{u+v}\frac{u+v'}{1+\frac{uv'}{c^2}}\frac{\Delta t}{\sqrt{1-\frac{(v'_{rel})^2}{c^2}}}$$

Final (Relativistic) Solution for Problem 2:

Calculating $T_{Walkway} - T_{Floor}$ gives:

$$\Delta T = \left[\frac{1}{\sqrt{1-\frac{v'^2}{c^2}}}\left(\frac{v'}{v}-1\right) - \frac{c^2+uv}{c^2+uv'}\left(\frac{u+v'}{u+v}\right) + \frac{c^2+uv'}{\sqrt{\left(c^2-v'^2\right)\left(c^2-u^2\right)}}\right]\Delta t$$

In the limit in which u, v, and v′ are all much less than c, this expression simplifies to:

$$\Delta T = \left[\frac{v'}{v} - \frac{u+v'}{u+v}\right]\Delta t$$

This is the non-relativistic solution to problem 2, so we can be pretty confident in our relativistic solution.

The expression for ΔT can probably (hopefully) be simplified some more.